Numerical study of natural convective heat transfer with large temperature differences

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Abstract *Steady-state two-dimensional solutions to the full compressible Navier-Stokes equations are computed for laminar convective motion of a gas in a square cavity with large horizontal temperature differences. No Boussinesq or low-Mach number approximations of the Navier-Stokes equations are used. Results for air are presented. The ideal-gas law is used and viscosity is given by Sutherland's law. An accurate low-Mach number solver is developed. Here an explicit third-order discretization for the convective part and a line-implicit central discretization for the acoustic part and for the diffusive part are used. The semi-implicit line method is formulated in multistage form. Multigrid is used as the acceleration technique. Owing to the implicit treatment of the acoustic and the diffusive terms, the stiffness otherwise caused by high aspect ratio cells is removed. Low Mach number stiffness is treated by a preconditioning technique. By a combination of the preconditioning technique, the semi-implicit discretization and the multigrid formulation a convergence behaviour is obtained which is independent of grid size, grid aspect ratio, Mach number and Rayleigh number. Grid converged results are shown for a variety of Rayleigh numbers.*

1. Introduction

Gas gaps between vertical parallel walls have been used for many decades to reduce heat transfer. Their use with large horizontal temperature differences has become increasingly important during the last three decades. Insulation using double walls, nuclear reactors, fire within buildings are only a few examples of applications.

Buoyancy-driven flows, especially in two dimensions, have been the object of thorough study for over 50 years. Most studies in the past have dealt with rectangular domains with different aspect ratios. Davis and Jones (1983) presented a study which resulted in a benchmark solution for the problem of a two-dimensional flow of a Boussinesq fluid in a square cavity, which is heated on the left, cooled on the right and insulated on the top and bottom boundaries. They used the stream function-vorticity formulation of the governing equations. Chenoweth and Paolucci (1986) investigated the steady-state flow in rectangular cavities with large temperature differences between vertical isothermal walls. They used the transient form of the flow equations, simplified for low-Mach number flows. Le Quéré (1991) studied incompressible flow in a thermally driven square cavity with a pseudo-spectral discretization scheme based on Chebyshev polynomials. Ramaswamy and Moreno (1997) computed three-dimensional buoyancy driven flows of incompressible fluids in complex

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geometries. In all the studies mentioned above rather coarse meshes were used.

In this paper solutions of the steady compressible full Navier-Stokes equations are computed. This means that no Boussinesq or low-Mach number approximations are used.

The computational method gives the solution very quickly and accurately, both on very fine meshes and on meshes with high grid aspect ratios. Normally the computational cost increases dramatically when finer meshes are used. Not only does the computational cost for one time step increase with the number of cells but also the number of time steps needed to obtain a steady state solution increases due to the Courant-Friedrich-Lewy restriction. Furthermore, very accurate solutions need high grid aspect ratio grids in zones with steep gradients. The use of such high grid aspect ratio meshes leads to unacceptably small time steps, so that often a choice has to be made between highly accurate solutions with enormous computational time and less accurate solutions with an acceptable computational time. In our method, the problem due to the grid aspect ratio is removed by the use of a line method. The low Mach number stiffness is avoided by an appropriate discretization and a local preconditioning technique. Multigrid is used as a convergence accelerator. The time needed for the calculation varies linearly with the number of grid cells.

Results are shown for the square cavity problem, for a variety of Rayleigh numbers.

2. Definition of the problem

We consider the flow in a differentially heated square cavity in which a temperature difference is applied to the vertical walls, while the horizontal walls are thermally insulated (Figure 1). This test case was the object of a previous comparison exercise for incompressible flow solvers with Boussinesq approximation (Davis and Jones, 1983), in which a series of reference solutions

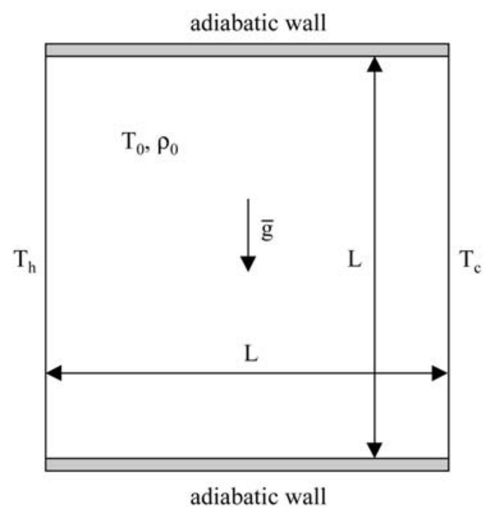

Figure 1.
Geometry, initial and boundary conditions for the thermally driven cavity problem

for Rayleigh numbers between 10^3 and 10^6 was produced. Here, we consider large temperature differences which impose the use of compressible solvers able to treat low Mach number flows.

For a compressible fluid, the Rayleigh number is defined as:

$$Ra = Pr\frac{g\rho_0^2(T_h - T_c)L^3}{T_0\mu_0^2},$$

where Pr is the Prandtl number (0.71 for air), μ the viscosity coefficient, g the gravitational constant, L the dimension of the square cavity, T_h and T_c respectively the hot and cold temperatures applied to the vertical walls, T_0 a reference temperature equal to $(T_h + T_c)/2$ and ρ_0 a reference density corresponding to T_0. The temperature difference may be represented by a non-dimensional parameter:

$$\varepsilon = \frac{T_h - T_c}{T_h + T_c}.$$

The heat transfer through the wall is represented by local and average Nusselt numbers Nu and $\overline{Nu}$:

$$Nu(y) = \frac{L}{k_0(T_h - T_c)} k\frac{\partial T}{\partial x}\bigg|_{\text{wall}}$$

$$\overline{Nu} = \frac{1}{L}\int_{y=0}^{y=L} Nu(y)dy$$

where $k(T)$ is the heat conduction coefficient, $k_0 = k(T_0)$. In the test cases considered here, the Prandtl number is assumed to remain constant, equal to 0.71, and the viscosity coefficient is given by Sutherland's law:

$$\frac{\mu(T)}{\mu^*} = \left(\frac{T}{T^*}\right)^{\frac{3}{2}}\frac{T^* + S}{T + S}, k(T) = \frac{\mu(T)C_p}{Pr},$$

where $T^* = 273$K, $S = 110.5$K, $\mu^* = 1.68\ 10^{-5}$kg/m/s, $C_p = \gamma R/(\gamma - 1)$, $\gamma = 1.4$ and $R = 287.0$J/kg/K. The influence of the temperature on C_p is of no consequence.

The problem is completely defined by the Rayleigh number, the value of ε, a reference state: $P_0 = 101{,}325$Pa, $T_0 = 600$K, $\rho_0 = P_0/(R\ T_0)$ and the initial conditions: $\forall(x,y) \in [0,L]^2$,

$$T(x,y) = T_0$$
$$\rho(x,y) = \rho_0$$
$$u(x,y) = v(x,y) = 0.$$

3. Computational method

3.1 Governing equations

The two-dimensional steady Navier-Stokes equations in conservative form for a compressible fluid are:

$$\frac{\partial F_c}{\partial x} + \frac{\partial F_a}{\partial x} + \frac{\partial G_c}{\partial y} + \frac{\partial G_a}{\partial y} = \frac{\partial F_v}{\partial x} + \frac{\partial G_v}{\partial y} + S,$$

where F_c and G_c are the convective fluxes, F_a and G_a are the acoustic fluxes and F_v and G_v are the viscous fluxes, in our method defined by:

$$F_c = \begin{bmatrix} 0 \\ \rho u^2 \\ \rho uv \\ 0 \end{bmatrix}, F_a = \begin{bmatrix} \rho u \\ p \\ 0 \\ \rho Hu \end{bmatrix}, F_v = \begin{bmatrix} 0 \\ \tau_{xx} \\ \tau_{xy} \\ u\tau_{xx} + v\tau_{xy} + k\frac{\partial T}{\partial x} \end{bmatrix},$$

$$G_c = \begin{bmatrix} 0 \\ \rho uv \\ \rho v^2 \\ 0 \end{bmatrix}, G_a = \begin{bmatrix} \rho v \\ 0 \\ p \\ \rho Hv \end{bmatrix}, G_v = \begin{bmatrix} 0 \\ \tau_{yx} \\ \tau_{yy} \\ u\tau_{yx} + v\tau_{yy} + k\frac{\partial T}{\partial y} \end{bmatrix}$$

where ρ is the density, u and v are the Cartesian components of velocity, p is the pressure, T is the temperature, H is the total enthalpy and τ_{ij} are the components of the viscous stress tensor.The source term S is given by

$$S = \begin{bmatrix} 0 \\ 0 \\ -\rho g \\ -\rho gv \end{bmatrix}$$

where g is the gravitational acceleration constant.

3.2 Discretization

We consider an orthogonal grid. The convective part of the equations is discretized with velocity upwinding:

$$F_{c_{i+1/2}} = u_{i+1/2}[0 \rho u \rho v 0]^T_{L/R},$$
$$G_{c_{j+1/2}} = v_{j+1/2}[0 \rho u \rho v 0]^T_{L/R},$$

where the left (L) and right (R) state variables are extrapolated with the third order Van Leer-κ method. The acoustic and viscous parts are discretized centrally. The source term is evaluated nodewise. Owing to the central discretization of the acoustic part, artificial dissipation for the pressure and temperature terms is necessary. A stabilization term is added to the mass flux. In the x-direction the mass flux is modified by

$$\frac{1}{2}\left[\frac{p_{i+1}-p_i}{\beta_x}+|u|\rho_T(T_{i+1}-T_i)\right],$$

where β_x and β_y have the dimension of velocity. A similar modification is used in the y-direction. We have taken

$$\beta_x = w_r + \frac{2\nu}{\Delta x}, \beta_y = w_r + \frac{2\nu}{\Delta y},$$

where w_r is a local velocity. Full details on the discretization are given in Vierendeels *et al.* (1999) .

3.3 Time marching method

Applying the pseudo-compressibility method to the Navier-Stokes equations gives:

$$\Gamma\frac{\partial Q}{\partial \tau}+\frac{\partial F_c}{\partial x}+\frac{\partial F_a}{\partial x}+\frac{\partial G_c}{\partial y}+\frac{\partial G_a}{\partial y}=RHS.$$

Q is the vector of the so-called viscous variables $Q=[p\ u\ v\ T]^T$,where T denotes the temperature and ^{T} the transposed vector. As preconditioning matrix Γ, a simplified form of Weiss and Smith's (1995) preconditioner is used, only suitable for low Mach number flows (Vierendeels *et al.*, 1999):

$$\Gamma = \begin{bmatrix} \Theta & 0 & 0 & \rho_T \\ 0 & \rho & 0 & 0 \\ 0 & 0 & \rho & 0 \\ \Theta H - 1 & 0 & 0 & 0 \end{bmatrix},$$

where $\Theta = \frac{1}{\beta^2} - \frac{\rho_T}{\rho C_p}$,$\rho_T$ is the derivative of ρ with respect to T,β has the dimension of velocity.

A multistage stepping with four stages is used:

$$\begin{aligned} Q^{(0)} &= Q^n \\ Q^{(1)} &= Q^{(0)} + \alpha_1 cfl\Delta Q^{(0)} \\ Q^{(2)} &= Q^{(0)} + \alpha_2 cfl\Delta Q^{(1)} \\ Q^{(3)} &= Q^{(0)} + \alpha_3 cfl\Delta Q^{(2)} \\ Q^{(4)} &= Q^{(0)} + \alpha_4 cfl\Delta Q^{(3)} \\ Q^{n+1} &= Q^{(4)}, \end{aligned}$$

with $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ equal to $\{1/4, 1/3, 1/2, 1\}$ and *cfl* set equal to 1.5. The $\Delta Q^{(m)}$ of each stage is given by $\Delta Q^{(m)} = Q^{(m+1)^*} - Q^{(m)}$, where $Q^{(m+1)^*}$ is calculated from:

$$\left(\frac{\Gamma}{\Delta\tau} + \frac{2}{\Delta x_i \Delta y_i}\left(\overline{A_v} + \overline{A_d}\right)\right)\left(Q^{(m+1)^*} - Q^{(m)}\right) + \frac{\partial F_c^{(m)}}{\partial x} + \frac{\partial F_a^{(m)}}{\partial x} - \frac{\partial F_d^{(m)}}{\partial x} + \frac{\partial G_c^{(m)}}{\partial y} + \frac{\partial G_a^{(m+1)^*}}{\partial y} - \frac{\partial G_d^{(m+1)^*}}{\partial y} - L^{(m),(m+1)^*}(Q) = S^{(m)},$$

for lines in the y-direction.

$L^{(m),(m+1)^*}(Q)$ is the discretized form of the operator (e.g. for lines in the y-direction):

$$L^{(m),(m+1)^*}(Q) = \frac{\partial}{\partial x}\left(R_{xx}\frac{\partial Q^{(m)}}{\partial x}\right) + \frac{\partial}{\partial x}\left(R_{xy}\frac{\partial Q^{(m)}}{\partial y}\right) + \frac{\partial}{\partial y}\left(R_{yx}\frac{\partial Q^{(m)}}{\partial x}\right) + \frac{\partial}{\partial y}\left(R_{yy}\frac{\partial Q^{(m+1)^*}}{\partial y}\right),$$

where R_{xx}, R_{xy}, R_{yx} and R_{yy} are given by

$$R_{xx} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{4}{3}\mu & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & \frac{4}{3}\mu u & \mu v & k \end{bmatrix}, \quad R_{xy} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2}{3}\mu & 0 \\ 0 & \mu & 0 & 0 \\ 0 & \mu v & -\frac{2}{3}\mu u & 0 \end{bmatrix},$$

$$R_{yx} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & -\frac{2}{3}\mu & 0 & 0 \\ 0 & -\frac{2}{3}\mu v & \mu u & 0 \end{bmatrix}, \quad R_{yy} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & \frac{4}{3}\mu & 0 \\ 0 & \mu u & \frac{4}{3}\mu v & k \end{bmatrix}$$

$\overline{A_v}$ is defined as

$$\overline{A_v} = \frac{1}{2}\left(R_{xx_{i+1/2}}\frac{\Delta y_{i+1/2}}{\Delta x_{i+1/2}} + R_{xx_{i-1/2}}\frac{\Delta y_{i-1/2}}{\Delta x_{i-1/2}}\right)$$

and $\overline{A_d}$ is defined as

$$\overline{A_d} = \frac{1}{2}\left(D_{x_{i+1/2}}\Delta y_{i+1/2} + D_{x_{i-1/2}}\Delta y_{i-1/2}\right)$$

where D_x is given by

$$D_x = \begin{bmatrix} \frac{1}{2\beta_x} & 0 & 0 & \frac{1}{2}|u|\rho_T \\ \frac{1}{2\beta_x}u & 0 & 0 & \frac{1}{2}|u|\rho_T u \\ \frac{1}{2\beta_x}v & 0 & 0 & \frac{1}{2}|u|\rho_T v \\ \frac{1}{2\beta_x}H & 0 & 0 & \frac{1}{2}|u|\rho_T H \end{bmatrix}$$

where β_x is given by

$$\beta_x = \sqrt{u^2 + v^2} + \frac{2\mu}{\rho \Delta x}. \qquad (1)$$

Similar expressions are used for lines in the y-direction. β is equal to β_x for lines in the x-direction and equal to β_y for lines in the y-direction.

Thus, a combination of an explicit method for the convective and source terms and an implicit line method for the acoustic, viscous and dissipation terms is used. The direction of the lines is alternated. This means that two multistage cycles are performed.

In the first one, $Q^{(m+1)^*}$ is calculated with lines in one direction and in the second one with lines in the other direction. This method has proven robustness for treating low Mach number flows on grids with high aspect ratios (Vierendeels *et al.*, 1999; Merci *et al.*, 2000).

The acoustic flux on level $m+1$ is written as

$$G_{a_{j+1/2}}^{m+1} = \begin{bmatrix} \rho^m v^{m+1} \\ 0 \\ p^{m+1} \\ \rho^m H^m v^{m+1} \end{bmatrix}_{j+1/2}.$$

For lines implicitly in the y-direction, the time step is computed by

$$\Delta\tau = \frac{1}{\frac{|u| + c_x}{\Delta x} + \frac{\omega|v|}{\Delta y}}, \qquad (2)$$

with $c_x = \sqrt{(u^2 + \beta^2)}$ and where ω is a scaling factor (Vierendeels *et al.*, 1999), set equal to 2.

In equation (1) there is no viscous contribution from the y-direction and in equation (2) there is no acoustic contribution from the y-direction, because these terms are treated implicitly in this direction.

The multistage semi-implicit method is accelerated with the multigrid technique. A full approximation scheme is used in a W-cycle with up to eight levels of grids. The computation is started on the finest grid in order to show the full performance of the multigrid method. For the restriction operator, full weighting is used. The prolongation is done with bilinear interpolation. Two pre- and post-relaxations are done, alternating the direction of the implicitly treated lines, as described above.

The pressure level at convergence is not imposed by the steady state equations. As a consequence, the mass content of the cavity is not imposed either. To obtain the correct mass content, given by the initial conditions, a correction step is done after each multigrid cycle. In this correction step the pressure in each node is multiplied by the factor f:

$$f = \frac{\text{initial mass}}{\text{current mass}}$$

During the convergence process of the flow problem, this factor converges to 1.

4. Validation

In order to validate the calculations, a comparison is made with the grid converged solution for $Ra = 10^6$, obtained by Le Quéré (1991) with the Boussinesq approximation. Therefore, a calculation was done with $Ra = 10^6$ and $\varepsilon = 0.01$. Results are computed on a 512×512 stretched grid, of which the maximum aspect ratio is 80. The calculated Nusselt number is 8.8257, which is very close to 8.8252, the value obtained by Le Quéré. This computation shows that, with a small value of ε, the result with the Boussinesq approximation is reproduced.

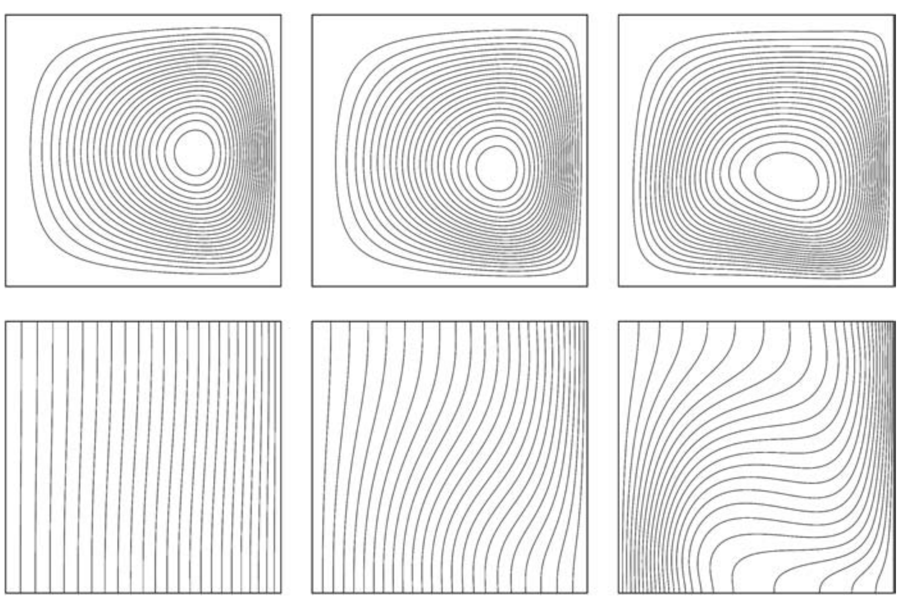

Figure 2.
Streamlines and temperature isolines for a viscous flow in a thermally driven cavity for $Ra = 10^2$, 10^3 and 10^4

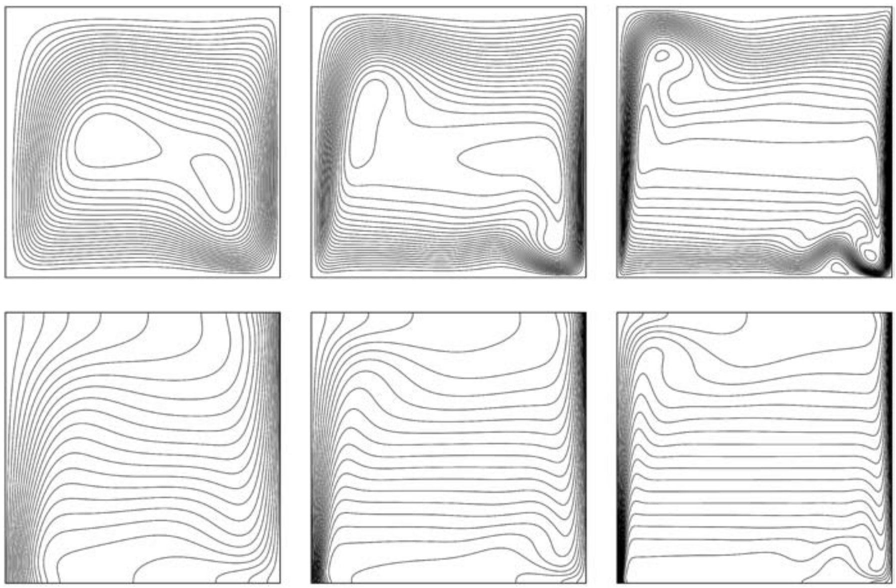

Figure 3. Streamlines and temperature isolines for a viscous flow in a thermally driven cavity for $Ra = 10^5$, 10^6 and 10^7

5. Results

For the present study, six Rayleigh numbers, $Ra = 10^2, 10^3, 10^4, 10^5, 10^6$ and 10^7, are considered with a temperature difference parameter $\varepsilon = 0.6$. Results are computed on a 512×512 stretched grid, of which the maximum aspect ratio is 80. Streamline patterns and temperature distributions are shown in Figures 2 and 3. Nusselt numbers and mean pressure values for the different Rayleigh numbers are given in Table I. The mean pressure is defined by

$$\bar{p} = \frac{1}{S}\int_S p dS$$

where S is the area of the cavity.

For the $Ra = 10^6$ and $Ra = 10^7$ case, a grid refinement study is performed. The results are summarized in Tables II-V. The extrapolated values are computed with Richardson's extrapolation method:

$$f_{\text{extrapol.}} = f_h - Ch^{\alpha},$$

Ra	$\overline{Nu}$	$\bar{p}/P_0$
1E2	0.9787	0.9574
1E3	1.108	0.9380
1E4	2.218	0.9146
1E5	4.480	0.9220
1E6	8.6866	0.92449
1E7	16.241	0.92263

Table I. Nusselt number and mean pressure for different Rayleigh numbers where

$$\alpha = \ln\left(\frac{f_h - f_{h'}}{f_{h/2} - f_{h'/2}}\right) / \ln(2)$$

and

$$C = \frac{f_h - f_{h/2}}{h^\alpha(1 - 2^{-\alpha})}.$$

Table II.
Nusselt numbers on the hot and cold wall and their mean value compared and extrapolated with Richardson's method (h = 1/1,024 and h′ = 1/768) for $Ra = 10^6$ for different grid sizes

N	$\overline{Nu}_h$	$\overline{Nu}_c$	$\overline{Nu}_{mean}$	% error
128	8.687033540	8.682329203	8.684681371	0.02192
256	8.686670690	8.685418776	8.686044733	0.00622
384	8.686618777	8.686050261	8.686334519	0.00288
512	8.686602534	8.686279239	8.686440887	0.00166
768	8.686591733	8.686446459	8.686519096	0.00076
1,024	8.686588147	8.686505978	8.686547062	0.00044
Rich. extrapol.	8.686584124	8.686585894	8.686585120	
α	–2.193965277	–1.940047977	–1.92215509	
C	1.00028E-12	–1.15481E-10	–6.22551E-11	

Table III.
Mean pressure extrapolated with Richardson's method (h = 1/1,024 and h′ = 1/768) for $Ra = 10^6$ for different grid sizes

N	$\bar{p}/P_0$	% error
128	0.924412352	0.00809
256	0.924466483	0.00224
384	0.924477677	0.00103
512	0.924481740	0.00059
768	0.924484706	0.00027
1,024	0.924485761	0.00015
Rich. extrapol.	0.924487176	
α	–1.941587882	
C	–2.02316E-12	

Table IV.
Nusselt numbers on the hot and cold wall and their mean value compared and extrapolated with Richardson's method (h = 1/1,024 and h′ = 1/768) for $Ra = 10^7$ for different grid sizes

N	$\overline{Nu}_h$	$\overline{Nu}_c$	$\overline{Nu}_{mean}$	% error
384	16.24078843	16.23691868	16.23885355	0.01319
512	16.24086990	16.23864746	16.23975868	0.00762
768	16.24093106	16.23992227	16.24042666	0.00350
1,024	16.24095331	16.24037979	16.24066655	0.00203
Rich. extrapol.	16.24098487	16.24100608	16.24099545	
α	–1.865009877	–1.913031113	–1.910839725	
C	–7.67201E-11	–1.09139E-09	–5.81926E-10	

h is given by $1/N$, where N is the number of grid cells in one direction. We used the method with $h = 1/1{,}024$ and $h' = 1/768$. The computed α values show that quadratic grid convergence is obtained. The error values show that, even for a small number of cells, good accuracy is obtained and that the computed results are correct up to 4 or 5 digits.

Figure 4 shows that the convergence behaviour is independent of the Rayleigh number. For $Ra = 10^6$ (Figure 5) and $Ra = 10^7$ (Figure 6), the convergence behaviour is shown for varying number of grid cells and grid aspect ratio. Both Figures show that there is no influence of number of grid cells and grid aspect ratio on the convergence behaviour.

6. Conclusion

A method of discretization of the low Mach number compressible Navier-Stokes equations is presented. The local preconditioning method is combined with a

N	$\bar{p}/P_0$	% error
382	0.922632009	0.00020
512	0.922632888	0.00011
768	0.922633472	0.00004
1,024	0.922633667	0.00002
Rich. extrapol.	0.922633887	
α	–2.187182255	
C	–5.71499E-14	

Table V. Mean pressure extrapolated with Richardson's method (h = 1/1,024 and h′ = 1/768) for $Ra = 10^7$ for different grid sizes

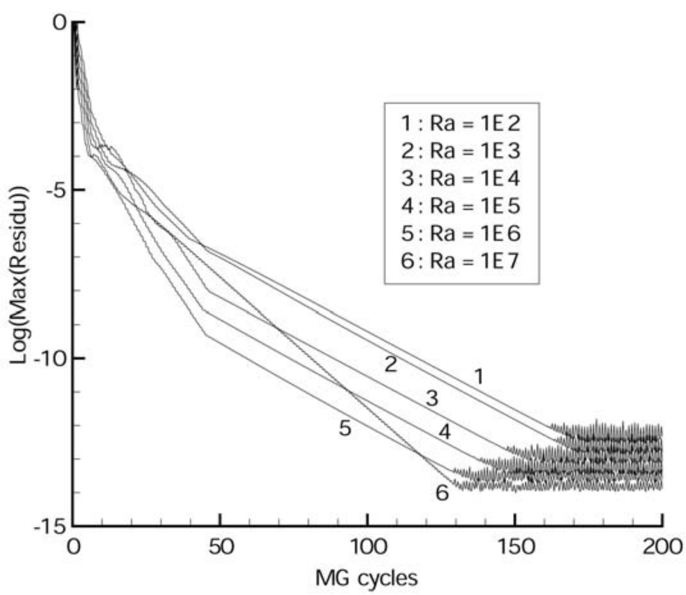

Figure 4. Convergence results for the thermally driven cavity flow problem for different Rayleigh numbers on a 512 × 512 grid with maximum grid aspect ratio equal to 80

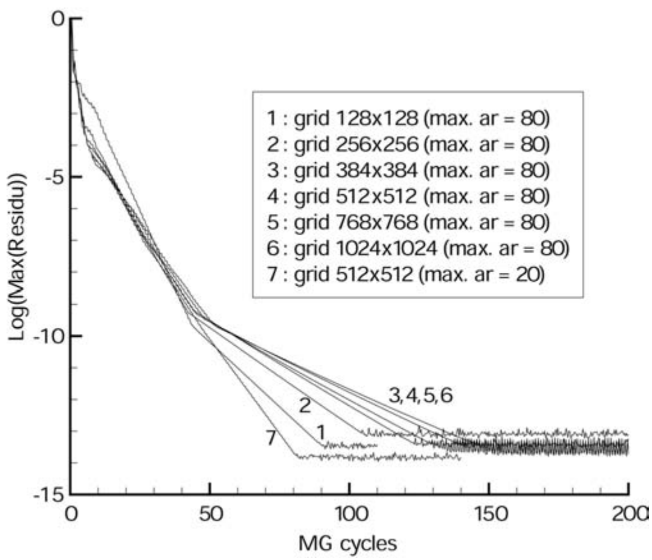

Figure 5.
Convergence results for the thermally driven cavity flow problem with $Ra = 10^6$ for different sizes of grid with different grid aspect ratios

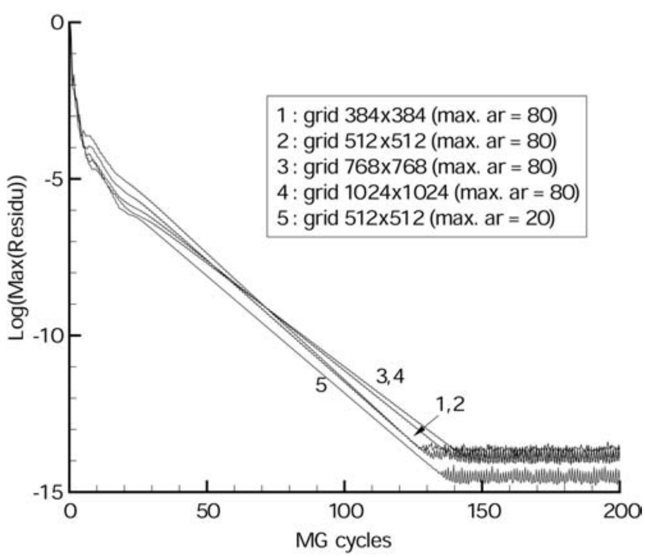

Figure 6.
Convergence results for the thermally driven cavity flow problem with $Ra = 10^7$ for different sizes of grid with different grid aspect ratios

line solver in order to remove the stiffness coming from high grid aspect ratios. This line solver is used in a multistage stepping scheme and accelerated with the multigrid method. The thermally driven cavity test case shows that the accuracy of the discretization method is very good. Quadratic grid convergence was obtained. The convergence of the solution method is very fast, independent of the Rayleigh number, the number of grid cells and the grid aspect ratio.

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